On Mellin convolution operators in Bessel potential spaces[1]

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Abstract

Mellin convolution equations acting in Bessel potential spaces are considered. The study is based upon two results. The first one concerns the interaction of Mellin convolutions and Bessel potential operators (BPOs). In contrast to the Fourier convolutions, BPOs and Mellin convolutions do not commute and we derive an explicit formula for the corresponding commutator in the case of Mellin convolutions with meromorphic symbols. These results are used in the lifting of the Mellin convolution operators acting on Bessel potential spaces up to operators on Lebesgue spaces. The operators arising belong to an algebra generated by Mellin and Fourier convolutions acting on $\mathbb{L}_p$-spaces. Fredholm conditions and index formulae for such operators have been obtained earlier by R. Duduchava and are employed here. Note that the results of the present work find numerous applications in boundary value problems for partial differential equations, in particular, for equations in domains with angular points.

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Introduction

Boundary value problems for elliptic equations in domains with angular points play an important role in applications and have a rich and exciting history. A prominent representative of this family is the Helmholtz equation. In the classical $\mathbb{W}^1$-setting, the existence and uniqueness of the solution of coercive systems with various type of boundary conditions are easily obtainable by using the Lax-Milgram Lemma (see, e.g., [23] where Laplace-Beltrami equations are considered on smooth surface with Lipschitz boundary). Similar problems arise in new applications in physics, mechanics and engineering. Thus recent publications on nano-photonics [1, 27] deal with physical and engineering problems described by BVPs for the Helmholts equation in $2D$ domains with angular points. They are investigated with the help of a modified Lax-Milgram Lemma for so called T-coercive operators. Similar problems occur for the Lamé systems in elasticity, Cauchy-Riemann systems, Carleman-Vekua systems in generalized analytic function theory etc.

Despite an impressive number of publications and ever growing interest to such problems, the results available to date are not complete. In particular, serious difficulties arise if an information on the solvability in non-classical setting in the Sobolev spaces $\mathbb{W}^1_p$, $1 < p < \infty$ is required, and one wants to study the solvability of equivalent boundary integral equations in the trace spaces $\mathbb{W}^{1-1/p}_p$ on the boundary. Integral equations arising in this case often have fixed singularities in the kernel and are of Mellin convolution type. For example, [2] describes how model BVP's in corners emerge from the localization of BVP for the Helmholtz equation in domains with Lipschitz boundary. Consequently, an attempt to study the corresponding Mellin convolution operators in Bessel potential spaces has been undertaken in [18]. However, not all of the results presented there are correct and one of the aims of this work is to provide correct formulations and proofs. We also hope that the results of the present paper will be helpful in further studies of boundary value problems for various elliptic equations in Lipschitz domains.

One such a model problem has been studied in [22]. More precisely, consider the following BVP with mixed Dirichlet–Neumann boundary conditions,

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0, & x \in \Omega_\alpha, \\ u^+(t) = g(t), & t \in \mathbb{R}^+, \\ (\partial_{\boldsymbol{\nu}} u)^+(t) = h(t), & t \in \mathbb{R}_\alpha, \end{cases} \tag{1}$$

in a corner Ω_α of magnitude α,

$$\partial\Omega_\alpha = \mathbb{R}^+ \cup \mathbb{R}_\alpha, \ \ \mathbb{R}^+ = (0, \infty),$$
$$\mathbb{R}_\alpha := \{t e^{i\alpha} = (t \cos\,\alpha, t \sin\,\alpha) \ : \ t \in \mathbb{R}^+\}.$$

By [22] the BVP (1) is reduced to the following equivalent system of boundary integral equations on $\mathbb{R}^+$,

$$\left\{\begin{aligned} &\varphi+\frac{1}{2\pi}\left[\boldsymbol{K}^1_{e^{i\alpha}}+\boldsymbol{K}^1_{e^{-i\alpha}}\right]\psi=G_1,\\ &\psi-\frac{1}{2\pi}\left[\boldsymbol{K}^1_{e^{i\alpha}}+\boldsymbol{K}^1_{e^{-i\alpha}}\right]\varphi=H_1\end{aligned}\right. \tag{2}$$

where

$$\boldsymbol{K}^1_{e^{\pm i\alpha}}\psi(t):=\frac{1}{\pi}\int_0^\infty\frac{\psi(\tau)d\tau}{t-e^{\pm i\alpha}\tau},\qquad 0<|\alpha|<\pi, \tag{3}$$

are Mellin convolution operators with homogeneous kernels of order -1 (see e.g. [15, 16] and Section 1 below), also called integral equations with fixed singularities in the kernel. Similar integral operators arise in the theory of singular integral equations with the complex conjugation if the contour of integration possess corner points. A complete theory of such equations is presented in [19, 20], whereas various approximation methods have been investigated in [8, 9, 10]. For a more detailed survey of this theory, applications in elasticity, and numerical methods for the corresponding equations we refer the reader to [12, 13, 15, 16, 33, 5, 6, 7]. Note that a similar approach has been employed by M. Costabel and E. Stephan [3, 4] in order to study boundary integral equations on curves with corner points.

Nevertheless, the results available are not sufficient in order to solve the problems arising in the investigation of BVP (1). First of all, we are looking for a solution to BVP (3) in the classical (finite energy) formulation

$$\begin{gathered} g\in\mathbb{H}^{1/2}(\mathbb{R}^+),\quad h\in\mathbb{H}^{-1/2}(\mathbb{R}_\alpha),\quad u\in\mathbb{H}^1(\Omega_\alpha)=\mathbb{W}^1(\Omega_\alpha),\\ u(x)=\mathcal{O}(1)\quad\text{as}\quad |x|\to\infty\,, \end{gathered} \tag{4}$$

or in the non-classical formulation

$$\begin{gathered} g\in\mathbb{W}^{1-1/p}_p(\mathbb{R}^+),\quad h\in\mathbb{W}^{-1/p}_p(\mathbb{R}_\alpha),\quad u\in\mathbb{H}^1_p(\Omega_\alpha)=\mathbb{W}^1_p(\Omega_\alpha),\\ u(x)=\mathcal{O}(1)\quad\text{as}\quad |x|\to\infty,\quad 1<p<\infty. \end{gathered} \tag{5}$$

The non-classical formulation is very helpful to explore the maximal smoothness of a solution to the BVP. This plays an important role in approximation methods and other applications..

The corresponding equivalent system of boundary integral equation (2) must be considered in the Bessel potential space $\widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+)$ in the case of classical setting (4) or in the Besov (Sobolev-Slobodeckii) space $\widetilde{\mathbb{W}}^{-1/p}(\mathbb{R}^+)$ in the case of non-classical setting (5). While doing so one encounters the three major tasks.

- In general, Mellin convolution operators are not bounded in neither Besov nor Bessel potential spaces. Therefore, in order to study equations (2) in the spaces of interest, one has to find a subclass of multipliers with the boundedness property.

- If boundedness criteria for the operators associated with equation (2) are available, one can lift this equation from the Besov or the Bessel potential space to a Lebesgue space.

- The lifted equations should be studied in the Lebesgue space.

A suitable class of Mellin convolution operators bounded in the Bessel potential spaces was proposed in [18]. These are Mellin convolutions with *admissible meromorphic kernels* (see (20) below). Having proved the boundedness result, one can study convolution equations in Bessel potential spaces. In particular, by lifting an equation with Mellin convolution operator $\mathfrak{M}^0_a$ with the help of Bessel potential operators $\mathbf{\Lambda}^s_+$ and $\mathbf{\Lambda}^{s-r}_-$, one obtains an equation in $\mathbb{L}_p$-space with the operator $\mathbf{\Lambda}^{s-r}_-\mathfrak{M}^0_a\mathbf{\Lambda}^{-s}_+$. However, the resulting operator $\mathbf{\Lambda}^{s-r}_-\mathfrak{M}^0_a\mathbf{\Lambda}^{-s}_+$ is neither Mellin nor Fourier convolution and in order to describe its properties, one first has to study the commutators of Bessel potential operators and Fourier convolutions with discontinuous symbols. As was already mentioned, this problem has been considered in [18], but not all of the results of that work are correct. Therefore, in Section 1 the commutator problem is discussed once again, and Theorem 3.2, Corollary 3.3 below provide correct formulae for the corresponding commutators.

The lifted operator $\mathbf{\Lambda}^{s-r}_-\mathfrak{M}_a\mathbf{\Lambda}^{-s}_+$ belongs to the Banach algebra generated by Mellin and Fourier convolution operators with discontinuous symbols. Such algebras have been studied before in [17] and the results obtained are systematized and updated in the recent paper [18]. In § 2, these results are applied to the lifted equation, hereby establishing properties of the initial Mellin convolution equation in the Bessel potential space.

The results of the present paper are applied to BVPs for the Helmholtz and Lamé equations in domains with corners and the corresponding paper of R. Duduchava, M. Tsaava and T. Tsutsunava will appear soon. These problems were investigated earlier only by means of Lax-Milgram Lemma [1]. In contrast to that, the approach of the present work is more fruitful and provides better tools to analyze the solvability of the equations involved and the asymptotic behaviour of their solutions. Moreover, it can also be used to study the Schrödinger operator on combinatorial and quantum graphs. Such a problem has attracted a lot of attention recently, since the operator mentioned has a wide range of applications in nano-structures [30, 31] and possesses interesting properties. Another area where the results of the present paper can be useful, is the study of Mellin pseudodifferential operators on graphs. This problem has been considered in [32] but in the periodic case only. Moreover, some of the result obtained play an important role in the theory of approximation methods for Mellin operators in Bessel potential spaces.

The present paper is organized as follows. In the first two sections we define Mellin convolution operators and recall some of their properties. In the second section we also consider Fourier convolution operators in the Bessel potential spaces and

discuss the lifting of these operators from the Bessel potential spaces to Lebesgue spaces, mostly according the papers [15, 24]. For Mellin convolutions such a lifting operation has not been studied before, and in Section 3 the interaction between Bessel potential operators and the Mellin convolution $\mathbf{K}^1_c$ with the kernel $(t-c\tau)^{-1}$ is considered. In particular, we derive formulae for commutators of Bessel potential operators and Mellin convolutions, and these results are crucial for our further considerations.

Section 4 recalls results from [17, 18] concerning the Banach algebra generated by Fourier and Mellin convolution operators in Lebesgue spaces with weight. These results, together with Theorem 3.2 and Corollary 3.3, are used in Section 5 in order to describe the lifting of Mellin convolution operators from the Bessel potential spaces up to operators in Lebesgue spaces. It turns out that the objects arising belong to a Banach algebra generated by Mellin and Fourier convolutions in $\mathbb{L}_p$-space on the semi-axis. The main result here is represented by Theorem 5.1 and Theorem 5.2, where the interaction between Bessel potential operators and the Mellin convolution resulting from the lifting of a model operator $\mathbf{K}^1_c$ is described. Theorem 5.3 deals with the lifting of the operator $\mathbf{K}^2_c$. In conclusion of Section 5, we present explicit formulae for the symbols of Mellin convolution operators with meromorphic kernels, which allow us to find Fredholm criteria and an index formula for the operators under consideration (see Theorem 5.4 and Corollary 5.6).

1 Mellin convolution operators

Equations (2) are a particular case of the Mellin convolution equation

$$\mathfrak{M}^0_a\varphi(t) := c_0\varphi(t) + \frac{c_1}{\pi i}\int_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \int_0^\infty \mathcal{K}\left(\frac{t}{\tau}\right)\varphi(\tau)\frac{d\tau}{\tau} = f(t) \tag{6}$$

where $c_0, c_1 \in \mathbb{C}$. If the kernel $\mathcal{K}$ satisfies the condition

$$\int_0^\infty t^\beta |\mathcal{K}(t)|\frac{dt}{t} < \infty, \quad 0 < \beta < 1,$$

then both equation (6) and analogous equations on the unit interval $I := (0,1)$ considered, respectively, on Lebesgue spaces $\mathbb{L}_p(\mathbb{R}^+)$ and $\mathbb{L}_p(I)$, are fully studied in [15].

Let a be an essentially bounded measurable $N \times N$ matrix function $a \in \mathbb{L}_\infty(\mathbb{R})$,

and let $\mathcal{M}_\beta$ and $\mathcal{M}_\beta^{-1}$ denote, respectively, the Mellin transform and its inverse, i.e.

$$\mathcal{M}_\beta\psi(\xi) := \int\limits_0^\infty t^{\beta-i\xi}\psi(t)\,\frac{dt}{t}, \quad \xi\in\mathbb{R},$$

$$\mathcal{M}_\beta^{-1}\varphi(t) := \frac{1}{2\pi}\int\limits_{-\infty}^\infty t^{i\xi-\beta}\varphi(\xi)\,d\xi, \quad t\in\mathbb{R}^+.$$

On the Schwartz space $\mathbb{S}(\mathbb{R}^+)$ of the fast decaying functions on $\mathbb{R}^+$, consider the following equation

$$\mathfrak{M}_a^0\varphi(t) = f(t), \tag{7}$$

where $\mathfrak{M}_a^0$ is the Mellin convolution operator,

$$\begin{aligned}\mathfrak{M}_a^0\varphi(t) :&= \mathcal{M}_\beta^{-1}a\mathcal{M}_\beta\varphi(t)\\ &= \frac{1}{2\pi}\int\limits_{-\infty}^\infty a(\xi)\int\limits_0^\infty\Big(\frac{t}{\tau}\Big)^{i\xi-\beta}\varphi(\tau)\,\frac{d\tau}{\tau}\,d\xi, \quad \varphi\in\mathbb{S}(\mathbb{R}^+).\end{aligned} \tag{8}$$

Note that equation (6) has the form (7) with the function a defined by

$$a(\xi) := c_0 + c_1\coth\pi(i\beta+\xi) + (\mathcal{M}_\beta\mathcal{K})(\xi).$$

Equations of the form (6), (7) and similar equations on finite intervals often arise in various areas of mathematics and mechanics (see [15, 28]).

The function $a(\xi)$ in (8) is usually referred to as the symbol of the Mellin operator $\mathfrak{M}_a^0$. Further, if the corresponding Mellin convolution operator $\mathfrak{M}_a^0$ is bounded on the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ endowed with the norm

$$\big\|\varphi \mid \mathbb{L}_p(\mathbb{R}^+, t^\gamma)\big\| := \Bigg[\int\limits_0^\infty t^\gamma|\varphi(t)|^p\,dt\Bigg]^{1/p},$$

then the symbol $a(\xi)$ is called a Mellin $\mathbb{L}_{p,\gamma}$–multiplier.

The two most important examples of Mellin convolution operators are

$$S_{\mathbb{R}^+}\varphi(t) := \frac{1}{\pi i}\int\limits_0^\infty\frac{\varphi(\tau)\,d\tau}{\tau-t}, \qquad \boldsymbol{K}_c^m\varphi(t) := \frac{1}{\pi i}\int\limits_0^\infty\frac{\tau^{m-1}\varphi(\tau)\,d\tau}{(t-c\,\tau)^m},$$

where $\operatorname{Im} c\neq 0$ and $m\in\mathbb{N}$ (see (3), (6)). The operator $S_{\mathbb{R}^+}$ is the celebrated Cauchy singular integral operator. The Mellin symbols of these operators are (cf. [18, § 2])

$$\sigma(S_{\mathbb{R}^+})(\xi) := -i\cot\pi(\beta-i\xi), \quad \xi\in\mathbb{R},$$

$$\sigma(\boldsymbol{K}_c^m)(\xi) := \binom{\beta-i\xi-1}{m-1}\frac{e^{\mp\pi(\beta-i\xi)i}i}{\sin\pi(\beta-i\xi)}\,c^{\beta-i\xi-m}, \quad 0<\pm\arg c<\pi,$$

where

$$\binom{\theta-1}{m-1} := \frac{(\theta-1)\cdots(\theta-m+1)}{(m-1)!}, \qquad \binom{\theta-1}{0} := 1.$$

In particular,

$$\mathcal{M}_\beta \mathcal{K}^1_{-c}(\xi) = \frac{c^{\,\beta-i\xi-1}}{\sin\pi(\beta-i\xi)}, \qquad -\pi < \arg c < \pi, \tag{9}$$

$$\mathcal{M}_\beta \mathcal{K}^1_{-1}(\xi) = \frac{1}{\sin\pi(\beta-i\xi)}, \qquad \xi \in \mathbb{R}. \tag{10}$$

The study of the equation (7) does not require much effort. The Mellin transform $\mathcal{M}_\beta$ converts (7) into the equation

$$a(\xi)\mathcal{M}_\beta\varphi(\xi) = \mathcal{M}_\beta f(\xi). \tag{11}$$

If $\inf|\det a(\xi)| > 0$ and the matrix-function a^{-1} is a Mellin $\mathbb{L}_{p,\gamma}$-multiplier, then equation (11) has the unique solution $\varphi = \mathcal{M}^0_{a^{-1}} f$.

The solvability of analogues of equation (8) on the unit interval $I = (0,1)$ in a weighted Lebesgue space $\mathbb{L}_p([0,1], t^\gamma)$ is also well understood. Thus if

$$1 < p < \infty, \qquad -1 < \gamma < p-1, \qquad \beta := \frac{1+\gamma}{p}, \qquad 0 < \beta < 1, \tag{12}$$

then one can use the isomorphisms

$$\begin{aligned} Z_\beta &: \mathbb{L}_p([0,1], t^\gamma) \to \mathbb{L}_p(\mathbb{R}^+), \qquad Z_\beta\varphi(\xi) := e^{-\beta\xi}\varphi(e^{-\xi}), \quad \xi \in \mathbb{R}^+, \\ Z_\beta^{-1} &: \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p([0,1], t^\gamma), \quad Z_\beta^{-1}\psi(t) := t^{-\beta}\psi(-\ln t), \quad t \in (0,1], \end{aligned} \tag{13}$$

and transform the corresponding equation on the unit interval I into an equivalent Wiener-Hopf equation, i.e. into the equation

$$W_{\mathcal{A}_\beta}\psi(x) = c_0\psi(x) + \int_0^\infty \mathcal{K}_1(x-y)\varphi(y)dy = f_0(t). \tag{14}$$

The Fourier transform of the kernel $\mathcal{K}_1$ is called the symbol of the corresponding Fourier convolution operator and is used to describe Fredholm properties, index and solvability of the equation (14). In passing note that Fourier convolution equations with discontinuous symbols are well studied [12, 13, 14, 15, 35].

2 Fourier convolution operators in the Bessel potential spaces: definition and lifting

Let N be a positive integer and let $\mathfrak{A}$ be a Banach algebra. If no confusion can arise, we write $\mathfrak{A}$ for both scalar and matrix $N \times N$ algebras with entries from $\mathfrak{A}$. Similarly, the same notation $\mathfrak{A}$ is used for the set of N-dimensional vectors with entries from $\mathfrak{A}$. It will be usually clear from the context what kind of space or algebra is considered.

Along with Mellin convolutions $\mathfrak{M}^0_a$, let us consider the Fourier convolution operators

$$W^0_a\varphi := \mathcal{F}^{-1}a\mathcal{F}\varphi, \quad \varphi \in \mathbb{S}(\mathbb{R}),$$

where $a \in \mathbb{L}_{\infty,loc}(\mathbb{R})$ is a locally bounded $N \times N$ matrix function, called the symbol of W^0_a and $\mathcal{F}$ and $\mathcal{F}^{-1}$ are, respectively, the direct and inverse Fourier transforms, i.e.

$$\mathcal{F}\varphi(\xi) := \int_{-\infty}^{\infty} e^{i\xi x}\varphi(x)\,dx, \quad \mathcal{F}^{-1}\psi(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\psi(\xi)\,d\xi, \quad x \in \mathbb{R}.$$

Let $1 < p < \infty$. An $N \times N$ matrix symbol $a(\xi)$, $\xi \in \mathbb{R}$ is called $\mathbb{L}_p$-multiplier if the corresponding convolution operator $W^0_a : \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R})$ is bounded. The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$. It is known (see, e.g. [15]), that $\mathfrak{M}_p(\mathbb{R})$ is a Banach subalgebra of $\mathbb{L}_\infty(\mathbb{R})$ which contains the algebra $\mathbf{V}_1(\mathbb{R})$ of all functions with finite variation. For $p = 2$ we have the exact equality $\mathfrak{M}_2(\mathbb{R}) = \mathbb{L}_\infty(\mathbb{R})$.

The operator

$$W_a := r_{\mathbb{R}^+}W^0_a : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+),$$

where $r_{\mathbb{R}^+} : \mathbb{L}_p(\mathbb{R}) \to \mathbb{L}_p(\mathbb{R}^+)$ denotes the restriction operator, is called the convolution on the semi-axis $\mathbb{R}^+$ or the Wiener-Hopf operator. It is worth noting that unlike the operators W^0_a and $\mathfrak{M}^0_a$, which possess the property

$$W^0_aW^0_b = W^0_{ab}, \quad \mathfrak{M}^0_a\mathfrak{M}^0_b = \mathfrak{M}^0_{ab} \quad \text{for all} \quad a, b \in \mathfrak{M}_p(\mathbb{R}), \tag{15}$$

the product of Wiener–Hopf operators cannot be computed by the simple rule (15). Thus for the operators W_a and W_b, a similar relation

$$W_aW_b = W_{ab} \tag{16}$$

is valid if and only if either $a(\xi)$ has an analytic extension into the lower half plane or $b(\xi)$ has an analytic extension into the upper half plane [15].

If conditions (12) hold, the isometrical isomorphisms (13) are extended to the following isomorphisms of Lebesgue spaces

$$Z_\beta : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \to \mathbb{L}_p(\mathbb{R}), \qquad Z_\beta\varphi(\xi) := e^{-\beta\xi}\varphi(e^{-\xi}), \quad \xi \in \mathbb{R},$$

$$Z^{-1}_\beta : \mathbb{L}_p(\mathbb{R}) \to \mathbb{L}_p(\mathbb{R}^+, t^\gamma), \quad Z^{-1}_\beta\psi(t) := t^{-\beta}\psi(-\ln t), \quad t \in \mathbb{R}^+,$$

and provide the following connection between the Fourier and Mellin transformations and the corresponding convolution operators–viz.,

$$\mathcal{M}_\beta = \mathcal{F}\mathbf{Z}_\beta, \qquad \mathcal{M}_\beta^{-1} = \mathbf{Z}_\beta^{-1}\mathcal{F}^{-1},$$
$$\mathfrak{M}_a^0 = \mathcal{M}_\beta^{-1} a \mathcal{M}_\beta = \mathbf{Z}_\beta^{-1}\mathcal{F}^{-1} a \mathcal{F}\mathbf{Z}_\beta = \mathbf{Z}_\beta^{-1} W_a^0 \mathbf{Z}_\beta.$$

These identities also justify the following assertion.

Proposition 2.1 ([15]) *Let $1 < p < \infty$ and $-1 < \gamma < p-1$. The class of Mellin $\mathbb{L}_{p,\gamma}$-multipliers does not depend on the parameter γ and coincides with the Banach algebra $\mathfrak{M}_p(\mathbb{R})$ of Fourier $\mathbb{L}_p$-multipliers.*

Corollary 2.2 ([15]) *A Mellin convolution operator $\mathfrak{M}_a^0 : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \to \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ of (8) is bounded if and only if $a \in \mathfrak{M}_p(\mathbb{R})$.*

For $s \in \mathbb{R}$ and $1 < p < \infty$, the Bessel potential space, known also as a fractional Sobolev space, is a subspace of the Schwartz space $\mathbb{S}'(\mathbb{R})$ of the distributions having the finite norm

$$\|\varphi|\mathbb{H}_p^s(\mathbb{R})\| := \left[\int_{-\infty}^{\infty} \left|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}(\mathcal{F}\varphi)(t)\right|^p dt\right]^{1/p} < \infty.$$

For the integer parameters $s = m \in \mathbb{N}$, space $\mathbb{H}_p^s(\mathbb{R})$ coincides with the Sobolev space $\mathbb{W}_p^m(\mathbb{R})$ endowed with an equivalent norm

$$\|\varphi|\mathbb{W}_p^m(\mathbb{R})\| := \left[\sum_{k=0}^{m} \int_{-\infty}^{\infty} \left|\frac{d^k\varphi(t)}{dt^k}\right|^p dt\right]^{1/p}.$$

If $s < 0$, one gets the space of distributions. Moreover, $\mathbb{H}_{p'}^{-s}(\mathbb{R})$ is the dual to the space $\mathbb{H}_p^s(\mathbb{R}^+)$, provided that $p' := \dfrac{p}{p-1}$, $1 < p < \infty$. Note that $\mathbb{H}_2^s(\mathbb{R})$ is a Hilbert space with the inner product

$$\langle \varphi, \psi \rangle_s = \int_{\mathbb{R}} (\mathcal{F}\varphi)(\xi)\overline{(\mathcal{F}\psi)(\xi)}(1+\xi^2)^s d\xi\,, \quad \varphi, \psi \in \mathbb{H}^s(\mathbb{R}).$$

By r_Σ we denote the operator restricting functions or distributions defined on $\mathbb{R}$ to the subset $\Sigma \subset \mathbb{R}$. Thus $\mathbb{H}_p^s(\mathbb{R}^+) = r_{\mathbb{R}^+}(\mathbb{H}_p^s(\mathbb{R}))$, and the norm in $\mathbb{H}_p^s(\mathbb{R}^+)$ is defined by

$$\|f|\mathbb{H}_p^s(\mathbb{R}^+\| = \inf_\ell \|\ell f|\mathbb{H}_p^s(\mathbb{R})\|,$$

where ℓf stands for any extension of f to a distribution in $\mathbb{H}_p^s(\mathbb{R})$.

Further, we denote by $\widetilde{\mathbb{H}}_p^s(\mathbb{R}^+)$ the (closed) subspace of $\mathbb{H}_p^s(\mathbb{R})$ which consists of all distributions supported in the closure of $\mathbb{R}^+$.

Note that $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ is always continuously embedded in $\mathbb{H}^s_p(\mathbb{R}^+)$ and for $s \in (1/p-1, 1/p)$ these two spaces coincide. Moreover, $\mathbb{H}^s_p(\mathbb{R}^+)$ may be viewed as the quotient-space $\mathbb{H}^s_p(\mathbb{R}^+) := \mathbb{H}^s_p(\mathbb{R})/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^-)$, $\mathbb{R}^- := (-\infty, 0)$.

If the Fourier convolution operator (FCO) on the semi-axis $\mathbb{R}^+$ with the symbol $a \in \mathbb{L}_{\infty,loc}(\mathbb{R})$ is bounded in the space setting

$$W_a := r_{\mathbb{R}^+} W^0_a \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+).$$

we say that W_a has order r and a is an $\mathbb{L}_p$ multiplier of order r. The set of all $\mathbb{L}_p$ multipliers of order r is denoted by $\mathfrak{M}^r_p(\mathbb{R})$. Let us mention another description of the space $\mathfrak{M}^r_p(\mathbb{R})$, viz. $a \in \mathfrak{M}^r_p(\mathbb{R})$ if and only if $\lambda^{-r}a \in \mathfrak{M}_p(\mathbb{R}) = \mathfrak{M}^0_p(\mathbb{R})$, where $\lambda^r(\xi) := (1+|\xi|^2)^{r/2}$.

Note, that FCOs are particular cases of pseudodifferential operators (ΨDOs).

Theorem 2.3 *Let $1 < p < \infty$. Then*

1. *For any $r, s \in \mathbb{R}$ and for any $\gamma \in \mathbb{C}$, $\operatorname{Im}\gamma > 0$, pseudodifferential operators $\mathbf{\Lambda}^r_\gamma := \mathbf{\Lambda}^r_{+\gamma}$ and $\mathbf{\Lambda}^r_{-\gamma}$ defined by*

$$\begin{aligned} \mathbf{\Lambda}^r_\gamma = W_{\lambda^r_\gamma} \ &: \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+), \\ \mathbf{\Lambda}^r_{-\gamma} = W_{\lambda^r_{-\gamma}} \ &: \ \mathbb{H}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+), \end{aligned} \tag{17}$$

   where $\lambda^r_{\pm\gamma}(\xi) := (\xi \pm \gamma)^r$, $\xi \in \mathbb{R}^+$, are isomorphisms between the corresponding spaces.

2. *For any operator $\mathbf{A} \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ of order r, the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) & \stackrel{\mathbf{A}}{\longrightarrow} & \mathbb{H}^{s-r}_p(\mathbb{R}^+) \\ \uparrow \mathbf{\Lambda}^{-s}_\gamma & & \downarrow \mathbf{\Lambda}^{s-r}_{-\gamma} \\ \mathbb{L}_p(\mathbb{R}^+) & \stackrel{\mathbf{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma}{\longrightarrow} & \mathbb{L}_p(\mathbb{R}^+). \end{array} \tag{18}$$

   Thus the diagram (18) provides an equivalent lifting of the operator $\mathbf{A}$ of order r up to the operator $\mathbf{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma \ : \ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ of order 0.

3. *Let $\mu, \nu \in \mathbb{R}$. If a is an $\mathbb{L}_p$-multiplier of order r, then for any complex numbers γ_1, γ_2 such that $\operatorname{Im}\gamma_j > 0$, $j = 1, 2$, the operator $\mathbf{\Lambda}^\mu_{-\gamma_1} W_a \mathbf{\Lambda}^\nu_{\gamma_2}$ is a Fourier convolution $W_{a_{\mu,\nu}}$ of order $r + \mu + \nu$,*

$$W_{a_{\mu,\nu}} \ : \ \widetilde{\mathbb{H}}^{s+\nu}_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r-\mu}_p(\mathbb{R}^+), \tag{19}$$

   with the symbol

$$a_{\mu,\nu}(\xi) := (\xi - \gamma_1)^\mu a(\xi)(\xi + \gamma_2)^\nu.$$

In particular, the lifting of the operator W_a up to the operator $\mathbf{\Lambda}^{s-r}_{-\gamma} W_a \mathbf{\Lambda}^{-s}_{\gamma}$ acting in the space $\mathbb{L}_p(\mathbb{R}^+)$ is FCO of order zero with the symbol

$$a_{s-r,-s}(\xi) = \lambda^{s-r}_{-\gamma}(\xi) a(\xi) \lambda^{-s}_{\gamma}(\xi) = \left(\frac{\xi-\gamma}{\xi+\gamma}\right)^{s-r} \frac{a(\xi)}{(\xi+\gamma)^r}\,.$$

4. *The Hilbert transform $\mathbf{K}^1_1 = iS_{\mathbb{R}^+} = W_{-\mathrm{i}\,\mathrm{sign}}$ is a Fourier convolution operator and*

$$\mathbf{\Lambda}^{s}_{-\gamma_1} \boldsymbol{K}^1_1 \mathbf{\Lambda}^{-s}_{\gamma_2} = W_{i\, g^s_{-\gamma_1,\gamma_2}\,\mathrm{sign}},$$

where

$$g^s_{-\gamma_1,\gamma_2}(\xi) := \left(\frac{\xi-\gamma_1}{\xi+\gamma_2}\right)^s.$$

Proof. For the proof of items $(i)-(iii)$ we refer the reader to [15, Lemma 5.1] and [21, 24]. The item (iv) is a consequence of $(ii)-(iii)$ (see [15, 18]). ■

Note that the operator equality in (19) is in fact a consequence of the relation (16).

3 Mellin convolution operators in the Bessel potential spaces–lifting

In contrast to the Fourier convolution operators the lifted Mellin convolution operator is not a Mellin convolution anymore. Moreover, there are Mellin convolution operators $\mathfrak{M}^0_{a_\beta}$ with symbols $a_\beta \in \mathfrak{M}_p(\mathbb{R})$ which are unbounded in the Bessel potential spaces. Thus in order to study the Mellin convolutions in the space of Bessel potentials, one has to address the boundedness problem first. To this end, a class of integral operators with admissible kernels was introduced in [18]. For the sake of simplicity, here we consider a lighter version of such kernels.

Definition 1 *The function $\mathcal{K}$ is called an* admissible meromorphic kernel *if it can be represented in the form*

$$\mathcal{K}(t) := \sum_{j=0}^{\ell} \frac{d_j}{t-c_j} + \sum_{j=\ell+1}^{N} \frac{d_j}{(t-c_j)^{m_j}}, \tag{20}$$

where $d_j, c_j \in \mathbb{C}$, $j = 0,1,\ldots,N$, $m_{\ell+1},\ldots,m_N \in \{2,3,\ldots\}$, and $0 < \alpha_k := |\arg c_k| \leqslant \pi$ for $k = \ell+1,\ldots,N$.

Note that the kernel $\mathcal{K}(t)$ has poles at the points $c_0, c_1, \ldots, c_N \in \mathbb{C}$.

Recall that boundary integral operators for BVPs in planar domains with corners have admissible kernels (see (2) and [15, 16, 18, 22]).

Theorem 3.1 ([18, Theorem 2.5 and Corollary 2.6]) *Let $1 < p < \infty$ and $s \in \mathbb{R}$. If $\mathcal{K}$ is an admissible kernel, then the Mellin convolution operator*

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+), \tag{21}$$

where $a_\beta = \mathcal{M}_\beta \mathcal{K}$, is bounded.

The next result is crucial to what follows. Note that a similar assertion appears in [18], but the proof contains fatal errors.

Theorem 3.2 *Let $s \in \mathbb{R}$, $c, \gamma \in \mathbb{C}$, $-\pi < \arg c \leqslant \pi$, $\arg c \neq 0$, $0 < \arg \gamma < \pi$ and $-\pi < \arg(c\,\gamma) < 0$. Then*

$$\mathbf{\Lambda}^s_{-\gamma} \mathbf{K}^1_c = c^{-s} \mathbf{K}^1_c \mathbf{\Lambda}^s_{-c\,\gamma}, \tag{22}$$

where $c^{-s} = |c|^{-s} e^{-s \arg(c)\, i}$.

Proof of Theorem 3.2. Taking into account the mapping properties of Bessel potential operators (17) and the mapping properties of a Mellin convolution operator with an admissible kernel (21), one observes that both operators

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma} \mathbf{K}^1_c \quad &: \quad \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{r-s}_p(\mathbb{R}^+), \\ \mathbf{K}^1_c \mathbf{\Lambda}^s_{-c\,\gamma} \quad &: \quad \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{r-s}_p(\mathbb{R}^+) \end{aligned} \tag{23}$$

are correctly defined and bounded for all $s \in \mathbb{R}$, $1 < p < \infty$, since $0 < \arg \gamma < \pi$ and $0 < -\arg(c\,\gamma) < \pi$.

On the other hand, let us note that the inverse superpositions $\mathbf{K}^1_c \mathbf{\Lambda}^s_\gamma$ and $\mathbf{\Lambda}^s_{c\,\gamma} \mathbf{K}^1_c$ are correctly defined only for $1/p - 1 < s < 1/p$ and $s = 1, 2, \ldots$.

For a smooth function with compact support $\varphi \in C^\infty_0(\mathbb{R}^+)$ and for $k = 1, 2, \ldots$ we can use integration by parts and obtain

$$\begin{aligned} \frac{d^k}{dt^k} \boldsymbol{K}^1_c \varphi(t) &= \frac{1}{\pi} \int_0^\infty \frac{d^k}{dt^k} \frac{1}{t - c\,\tau} \varphi(\tau)\, d\tau = \frac{(-c)^{-k}}{\pi} \int_0^\infty \frac{d^k}{d\tau^k} \frac{1}{t - c\,\tau} \varphi(\tau)\, d\tau = \\ &= \frac{(-c)^{-k}}{\pi} \int_0^\infty \frac{1}{t - c\,\tau} \frac{d^k \varphi(\tau)}{d\tau^k}\, d\tau = (-c)^{-k} \Big(\boldsymbol{K}^1_c \frac{d^k}{dt^k} \varphi \Big)(t). \end{aligned} \tag{24}$$

Let us consider the case where s is a positive integer, i.e. $s = m = 1, 2, \ldots$. The Bessel potentials $\mathbf{\Lambda}^m_{\pm} = W_{\lambda^m_{\pm\gamma}}$ are the Fourier convolutions of order m and they represent ordinary differential operators of the order m, namely,

$$\mathbf{\Lambda}^m_{\pm\gamma} = W_{\lambda^m_{\pm\gamma}} = \Big(i\,\frac{d}{dt} \pm \gamma\Big)^m = \sum_{k=0}^{m} \binom{m}{k} i^k (\pm\gamma)^{m-k}\,\frac{d^k}{dt^k}\,. \tag{25}$$

By the relations (17) the mappings

$$\mathbf{\Lambda}^m_{\gamma} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}^{s-m}_p(\mathbb{R}^+),$$
$$\mathbf{\Lambda}^m_{-\gamma} : \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-m}_p(\mathbb{R}^+),$$

are isomorphisms of the corresponding spaces if $\operatorname{Im}\gamma > 0$.

Taking into account formulae (24) and (25), one obtains the relation

$$\begin{aligned}
\mathbf{\Lambda}^m_{\gamma}\mathbf{K}^1_c\varphi &= \Big(i\,\frac{d}{dt} + \gamma\Big)^m \mathbf{K}^1_c\varphi = \sum_{k=0}^{m} \binom{m}{k} i^k \gamma^{m-k}\,\frac{d^k}{dt^k}\,\mathbf{K}^1_c\varphi \\
&= \sum_{k=0}^{m} \binom{m}{k} i^k \gamma^{m-k} c^{-k}\Big(\mathbf{K}^1_c\,\frac{d^k}{dt^k}\,\varphi\Big)(t) = \\
&= c^{-m}\mathbf{K}^1_c\bigg(\sum_{k=0}^{m} \binom{m}{k} i^k\,(c\,\gamma)^{m-k}\,\frac{d^k}{dt^k}\,\varphi\bigg)(t) = \\
&= c^{-m}\mathbf{K}^1_c\mathbf{\Lambda}^m_{c\gamma}\varphi, \qquad \varphi \in \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+).
\end{aligned}$$

Thus for $s = m = 1, 2, \ldots$, formula (22) is proved.

If s is a negative integer, $s = -1, -2, \ldots = -m$, formulae (22) can be established by applying the inverse operators and $\mathbf{\Lambda}^{-m}_{\gamma}$ and $\mathbf{\Lambda}^{-m}_{-c\gamma}$, respectively, from the left and from the right to the already proven operator equality

$$\mathbf{\Lambda}^m_{\gamma}\mathbf{K}^1_c = c^{-m}\mathbf{K}^1_c\mathbf{\Lambda}^m_{c\gamma}, \quad m = 1, 2, \ldots\,.$$

Thus one obtains

$$\mathbf{K}^1_c\mathbf{\Lambda}^{-m}_{c\gamma} = c^{-m}\mathbf{\Lambda}^{-m}_{\gamma}\mathbf{K}^1_c \quad \text{or} \quad \mathbf{\Lambda}^{-m}_{\gamma}\mathbf{K}^1_c = c^{m}\mathbf{K}^1_c\mathbf{\Lambda}^{-m}_{c\gamma}$$

and for a negative $s = -1, -2, \ldots$, relation (22) is also proved.

In order to establish formula (22) for non-integer values of s, we can confine ourselves to the case $-1 < s < 0$. Indeed, any non-integer value $s \in \mathbb{R}$ can be represented in the form $s = s_0 + m$, where $-1 < s_0 < 0$ and m is an integer. Therefore, if for $s = s_0 + m$ the operators in (23) are correctly defined and bounded, and if the relations in question are valid for $-1 < s_0 < 0$, then we can write

$$\begin{aligned}
\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = \mathbf{\Lambda}^{s_0+m}_{-\gamma}\mathbf{K}^1_c &= c^{-m}\mathbf{\Lambda}^{s_0}_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^m_{-c\gamma} = c^{-s_0-m}\mathbf{K}^1_c\mathbf{\Lambda}^{s_0}_{-c\gamma}\mathbf{\Lambda}^m_{-c\gamma} \\
&= c^{-s_0-m}\mathbf{K}^1_c\mathbf{\Lambda}^{s_0+m}_{-c\gamma} = c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\gamma}.
\end{aligned}$$

Thus let us assume that $-1 < s < 0$ and consider the case $0 < \arg c < \pi$. Changing the orders of integration, we obtain

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) &= \frac{1}{2\pi^2}\, r_+ \int\limits_{-\infty}^{\infty} e^{-i\xi t}(\xi-\gamma)^s \int\limits_0^{\infty} e^{i\xi y} \int\limits_0^{\infty} \frac{\varphi(\tau)}{y-c\tau}\, d\tau\, dy\, d\xi \\ &= \frac{1}{2\pi^2}\, r_+ \int\limits_0^{\infty} \varphi(\tau) \int\limits_0^{\infty} \frac{1}{y-c\tau} \int\limits_{-\infty}^{\infty} e^{i\xi(y-t)}(\xi-\gamma)^s d\xi\, dy\, d\tau, \end{aligned} \tag{26}$$

where r_+ is the restriction to $\mathbb{R}^+$. In order to study the expression in the right-hand side of (26), one can use a well known formula

$$\int\limits_{-\infty}^{\infty} (\beta+ix)^{-\nu} e^{-ipx}\, dx = \begin{cases} 0 & \text{for} \quad p>0, \\ -\dfrac{2\pi(-p)^{\nu-1}e^{\beta p}}{\Gamma(\nu)} & \text{for} \ \ p<0, \end{cases}$$

$$\operatorname{Re}\nu > 0, \qquad \operatorname{Re}\beta > 0,$$

[26, Formula 3.382.6]. It can be rewritten in a more convenient form–viz.,

$$\int\limits_{-\infty}^{\infty} e^{i\mu\,\xi}(\xi-\gamma)^s\, d\xi = \begin{cases} 0 & \text{if} \ \ \mu<0,\ \operatorname{Im}\gamma>0, \\ \dfrac{2\pi\,\mu^{-s-1}e^{-\frac{\pi}{2}si+\mu\,\gamma i}}{\Gamma(-s)} & \text{if} \ \ \mu>0,\ \operatorname{Im}\gamma>0. \end{cases} \tag{27}$$

Applying (27) to the last integral in (26), one obtains

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) &= \frac{e^{-\frac{\pi}{2}si}}{\pi\Gamma(-s)} r_+ \int\limits_0^{\infty} \varphi(\tau)\, d\tau \int\limits_t^{\infty} \frac{e^{i(y-t)\gamma}dy}{(y-t)^{1+s}(y-c\tau)} \\ &= \frac{e^{-\frac{\pi}{2}si}}{\pi\Gamma(-s)} r_+ \int\limits_0^{\infty} \varphi(\tau)\, d\tau \int\limits_0^{\infty} \frac{e^{i\gamma\, y}dy}{y^{1+s}(y+t-c\tau)}. \end{aligned} \tag{28}$$

Let us also use the formula [26, Formula 3.383.10],

$$\int\limits_0^{\infty} \frac{x^{\nu-1}e^{-\mu\, x}\, dx}{x+\beta} = \beta^{\nu-1}e^{\beta\,\mu}\Gamma(\nu)\Gamma(1-\nu,\beta\mu), \tag{29}$$

$$\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0, \quad |\arg\beta| < \pi,$$

and represent the operator (28) in the form

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) = \frac{e^{-\frac{\pi}{2}si}}{\pi} r_+ \int\limits_0^{\infty} \frac{e^{-i\gamma(t-c\tau)}\Gamma(1+s,-i\gamma(t-c\tau))\varphi(\tau)\, d\tau}{(t-c\tau)^{1+s}}. \tag{30}$$

Consider now the inverse composition $\mathbf{K}_c^1 \mathbf{\Lambda}_{-c\,\gamma}^s \varphi(t)$. Changing the order of integration in the corresponding expression, one obtains

$$\begin{aligned}
\mathbf{K}_c^1 \mathbf{\Lambda}_{-c\,\gamma}^s \varphi(t) &:= \frac{1}{2\pi^2}\, r_+ \int_0^\infty \frac{1}{t - c\,y} \int_{-\infty}^\infty e^{-i\xi\, y} (\xi - c\,\gamma)^s \int_0^\infty e^{i\xi\,\tau} \varphi(\tau) d\tau\, d\xi\, dy \\
&= \frac{1}{2\pi^2} r_+ \int_0^\infty \varphi(\tau) \int_0^\infty \frac{1}{t - c\,y} \int_{-\infty}^\infty e^{i\xi(\tau - y)} (\xi - c\,\gamma)^s d\xi\, dy\, d\tau. \qquad (31)
\end{aligned}$$

In order to compute the expression in the right-hand side of (31), let us recall Formula 3.382.7 of [26],

$$\int_{-\infty}^\infty (\beta - ix)^{-\nu} e^{-ipx}\, dx = \begin{cases} 0 & \text{for} \quad p < 0, \\ \dfrac{2\pi\, p^{\nu-1} e^{-\beta\, p}}{\Gamma(\nu)} & \text{for} \quad p > 0, \end{cases}$$

$$\mathrm{Re}\,\nu > 0, \quad \mathrm{Re}\beta > 0,$$

and rewrite it in a form more suitable for our consideration–viz.,

$$\int_{-\infty}^\infty e^{i\mu\,\xi} (\xi + \omega)^s\, d\xi = \begin{cases} 0 & \mu > 0,\ \mathrm{Im}\,\omega > 0, \\ \dfrac{2\pi\, (-\mu)^{-s-1} e^{\frac{\pi}{2} si - \mu\,\omega i}}{\Gamma(-s)} & \mu < 0,\ \mathrm{Im}\,\omega > 0, \end{cases} \qquad (32)$$

$$\mathrm{Re}\, s < 0, \quad \mu \in \mathbb{R}, \quad \omega,\, s \in \mathbb{C}.$$

Using (32), we represent (31) in the form

$$\begin{aligned}
\mathbf{K}_c^1 \mathbf{\Lambda}_{-c\,\gamma}^s \varphi(t) &= \frac{e^{\frac{\pi}{2} si}}{\pi \Gamma(-s)}\, r_+ \int_0^\infty \varphi(\tau)\, d\tau \int_\tau^\infty \frac{e^{-ic\,\gamma(y-\tau)}\, dy}{(y - \tau)^{s+1} (t - c\,y)} \\
&= -\frac{e^{\frac{\pi}{2} si}}{\pi c \Gamma(-s)} r_+ \int_0^\infty \varphi(\tau)\, d\tau \int_0^\infty \frac{e^{-ic\gamma\, y}\, dy}{y^{s+1} (y - c^{-1} t + \tau)},
\end{aligned}$$

and the application of formula (29) leads to the representation

$$\begin{aligned}
\mathbf{K}_c^1 \mathbf{\Lambda}_{-c\,\gamma}^s \varphi(t) &= -\,\frac{c^{-1} e^{\frac{\pi}{2} si}}{\pi} r_+ \int_0^\infty \frac{e^{-ic\gamma(c^{-1}t - \tau)} \Gamma(1 + s, -ic\gamma(c^{-1}t - \tau)) \varphi(\tau)\, d\tau}{(\tau - c^{-1}t)^{1+s}} \\
&= \frac{c^s e^{-\frac{\pi}{2} si}}{\pi} r_+ \int_0^\infty \frac{e^{-i\gamma(t - c\,\tau)} \Gamma(1 + s, -i\gamma(t - c\,\tau)) \varphi(\tau)\, d\tau}{(t - c\,\tau)^{1+s}}. \qquad (33)
\end{aligned}$$

Now the relations (30) and (33) imply the equality (22) for $0 < \arg c < \pi$.

In the case $\operatorname{Im} c = 0$, $c < 0$, we proceed as for $0 < \arg c < \pi$ and arrive at the formula

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_{-1}\varphi(t) = \frac{e^{-\frac{\pi}{2}si}}{\pi} r_+ \int_0^\infty \frac{e^{-i\gamma(t+|c|\tau)}\Gamma(1+s, -i\gamma(t+|c|\tau))\varphi(\tau)\,d\tau}{(t+|c|\tau)^{1+s}}, \tag{34}$$

which is similar to (30). Further, instead of (33) we get

$$\mathbf{K}^1_{-1}\mathbf{\Lambda}^s_{\gamma}\varphi(t) = \frac{|c|^s e^{\frac{\pi}{2}si}}{\pi} r_+ \int_0^\infty \frac{e^{-i\gamma(t+|c|\tau)}\Gamma(1+s, -i\gamma(t+|c|\tau))\varphi(\tau)\,d\tau}{(t+|c|\tau)^{1+s}}, \tag{35}$$

and the relations (34) and (35) lead to the equality (22) for $\operatorname{Im} c = 0$, $c < 0$. □

Corollary 3.3 *Let* $0 < |\arg c| \leqslant \pi$, $\arg c \neq 0$, $0 < \arg\gamma < \pi$ *and* $-\pi < \arg(c\,\gamma) < 0$*. Then for arbitrary* $\gamma_0 \in \mathbb{C}$ *such that* $0 < \arg \gamma_0 < \pi$ *and* $-\pi < \arg(c\,\gamma_0)| < 0$*, one has*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s} W_{g_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma_0}, \tag{36}$$

where

$$g^s_{-\gamma,-\gamma_0}(\xi) := \left(\frac{\xi-\gamma}{\xi-\gamma_0}\right)^s. \tag{37}$$

If, in addition, $1 < p < \infty$ *and* $1/p - 1 < r < 1/p$ *then equality* (36) *can be supplemented as follows*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}\left[\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} + \mathbf{T}\right]\mathbf{\Lambda}^s_{-c\,\gamma_0}, \tag{38}$$

where $\mathbf{T} : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$ *is a compact operator, and if* c *is a real negative number, then* $c^{-s} := |c|^{-s}e^{-\pi si}$*.*

Proof. It follows from equalities (16) and (22) that

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = \mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{-\gamma_0}\mathbf{\Lambda}^s_{-\gamma_0}\mathbf{K}^1_c = c^{-s}W_{g_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma_0}$$

and (36) is proved. If $1 < p < \infty$ and $1/p - 1 < r < 1/p$, then the commutator

$$\mathbf{T} := W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c - \mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} \ : \ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$$

of Mellin and Fourier convolution operators is correctly defined and bounded. It is compact for $r = 0$ and all $1 < p < \infty$ (see [11, 17]). Due to Krasnoselsky interpolation theorem (see [29] and also [36, Sections 1.10.1 and 1.17.4]), the operator $\mathbf{T}$ is compact in all $\mathbb{L}_r$-spaces for $1/p - 1 < r < 1/p$. Therefore, the equality (36), can be rewritten as

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}\left[\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} + \mathbf{T}\right]\mathbf{\Lambda}^s_{-c\,\gamma_0}\,,$$

and we are done ■

Remark 1 *The assumption $1/p - 1 < r < 1/p$ in (38) cannot be relaxed. Indeed, the operator $W_{g^s_{-\gamma,-\gamma_0}} \mathbf{K}^1_c = \mathbf{\Lambda}^s_{-\gamma} \mathbf{\Lambda}^{-s}_{-\gamma_0} \mathbf{K}^1_c \; : \; \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$ is bounded for all $r \in \mathbb{R}$ (see (23)). But the operator $\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} \; : \; \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$ is bounded only for $1/p - 1 < r < 1/p$ because the function $g^s_{-\gamma,-\gamma_0}(\xi)$ has an analytic extension into the lower half-plane but not into the upper one.*

4 Algebra Generated by Mellin and Fourier Convolution Operators

In the present section we recall some results on Banach algebra, generated by Fourier and Mellin convolution operators in the Lebesgue space with weight from [17], revised in [18]. The exposition follows [18, Section 2]. For more general algebras we refer the reader to [17] and to [11, 35].

Let us consider the Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$ generated by Mellin convolution and Fourier convolution operators in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$. In particular, this algebra contains the operators

$$\mathbf{A} := \sum_{j=1}^{m} \mathfrak{M}^0_{a_j} W_{b_j}, \tag{39}$$

and their compositions. Here $\mathfrak{M}^0_{a_j}$ are Mellin convolution operators with continuous $N \times N$ matrix symbols $a_j \in C\mathfrak{M}_p(\overline{R})$, W_{b_j} are Fourier convolution operators with $N \times N$ matrix symbols $b_j \in C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^- \cup \overline{\mathbb{R}}^+)$. The algebra of $N \times N$ matrix $\mathbb{L}_p$-multipliers $C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\})$ consists of those piecewise-continuous $N \times N$ matrix multipliers $b \in \mathfrak{M}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$ which are continuous on the semi-axes $\mathbb{R}^-$ and $\mathbb{R}^+$ but might have finite jump discontinuities at 0 and at the infinity.

Note that the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ is actually a subalgebra of the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ generated by the Fourier convolution operators W_a with piecewise-constant symbols $a(\xi)$ in the space $\mathbb{L}_p(\mathbb{R}^+)$. Let $\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denote the ideal of all compact operators in $\mathbb{L}_p(\mathbb{R}^+)$. Since in the scalar case $N = 1$ the quotient algebra $\mathfrak{F}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative, the following proposition is true.

Proposition 4.1 ([17] and [18, Corollary 3.10]) *If $N = 1$, then the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative.*

To describe the symbol of the operator $\boldsymbol{A}$ of (39), consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma^-_2 \cup \Gamma^+_2 \cup \Gamma_3$, where (cf. Figure 1)

$$\Gamma_1 := \overline{\mathbb{R}} \times \{+\infty\}, \;\; \Gamma^{\pm}_2 := \{\pm\infty\} \times \overline{\mathbb{R}}^+, \;\; \Gamma_3 := \overline{\mathbb{R}} \times \{0\}.$$

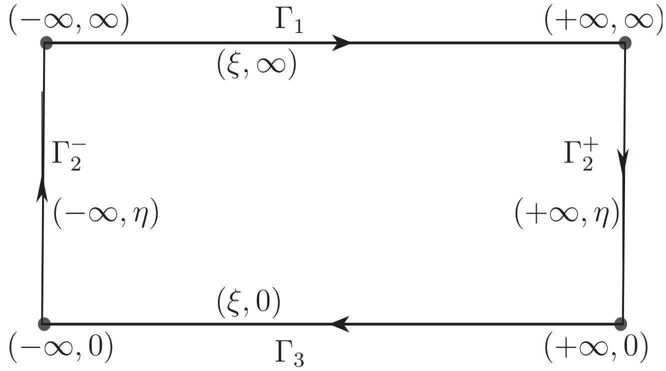

Figure 1: The domain $\mathfrak{R}$ of definition of the symbol $\mathcal{A}_p(\xi,\eta)$.

The symbol $\mathcal{A}_p(\omega)$ of the operator $\boldsymbol{A}$ in (39) is a function on the set $\mathfrak{R}$, viz.

$$\mathcal{A}_p(\omega) := \begin{cases} \sum_{j=1}^{m} a_j(\xi)(b_j)_p(\infty,\xi), & \omega=(\xi,\infty)\in\overline{\Gamma}_1, \\ \sum_{j=1}^{m} a_j(+\infty)b_j(-\eta), & \omega=(+\infty,\eta)\in\Gamma_2^+, \\ \sum_{j=1}^{m} a_j(-\infty)b_j(\eta), & \omega=(-\infty,\eta)\in\Gamma_2^-, \\ \sum_{j=1}^{m}(a_j)_p(\infty,\xi)(b_j)_p(0,\xi), & \omega=(\xi,0)\in\overline{\Gamma}_3. \end{cases} \tag{40}$$

In (40) for a piecewise continuous function $g\in PC(\overline{\mathbb{R}})$ we use the notation

$$\begin{aligned} g_p(\infty,\xi) &:= \frac{1}{2}\big[g(+\infty)+g(-\infty)\big]- \\ &\qquad -\frac{i}{2}\big[g(+\infty)-g(-\infty)\big]\cot\pi\Big(\frac{1}{p}-i\xi\Big), \\ g_p(t,\xi) &:= \frac{1}{2}\big[g(t+0)+g(t-0)\big]- \\ &\qquad -\frac{i}{2}\big[g(t+0)-g(t-0)\big]\cot\pi\Big(\frac{1}{p}-i\xi\Big), \end{aligned} \tag{41}$$

where $t,\xi\in\mathbb{R}$.

Arc condition ([25, 37]): The function $g_p(\infty,\xi)$ connects the point $g(-\infty)$ with $g(+\infty)$. More precisely, it fills up the discontinuity of the function g at ∞ with an oriented arc of the circle such that from every point of the arc the oriented interval $[g(-\infty),g(+\infty)]$ is seen under the angle π/p. Moreover, the oriented arc lies on the

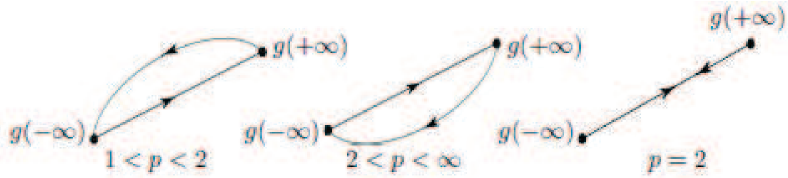

Figure 2: Arc condition.

left of the oriented interval if $1/2 < 1/p < 1$ (i.e., if $1 < p < 2$) and the oriented arc is on the right of the oriented interval if $0 < 1/p < 1/2$ (i.e., if $2 < p < \infty$). For $p = 2$ the oriented arc coincides with the oriented interval (see Figure 2).

A similar geometric interpretation is valid for the function $g_p(t,\xi)$, which connects the points $g(t-0)$ and $g(t+0)$ when g has a jump at $t \in \overline{\mathbb{R}}$.

The image of the function $\det \mathcal{A}_p(\omega)$, $\omega \in \mathfrak{R}$ is a closed curve in the complex plane. It follows from the continuity of the symbol at the angular points of the rectangle $\mathfrak{R}$ where the one-sided limits coincide. Thus

$$\mathcal{A}_p(\pm\infty,\infty) = \sum_{j=1}^{m} a_j(\pm\infty) b_j(\mp\infty),$$
$$\mathcal{A}_p(\pm\infty,0) = \sum_{j=1}^{m} a_j(\pm\infty) b_j(0\mp 0).$$

Hence, if the symbol of the corresponding operator is elliptic, i.e. if

$$\inf_{\omega\in\mathfrak{R}} \left|\det \mathcal{A}_p(\omega)\right| > 0,$$

the increment of the argument $(1/2\pi)\arg \mathcal{A}_p(\omega)$ when ω ranges through $\mathfrak{R}$ in the direction of orientation, is an integer. It is called the winding number or the index of the curve $\Gamma := \{z \in \mathbb{C} : z = \det \mathcal{A}_p(\omega),\ \omega \in \mathfrak{R}\}$ and is denoted by $\operatorname{ind}\det \mathcal{A}_p$.

Theorem 4.2 ([18, Theorem 3.13]) *Let $1 < p < \infty$ and let $\mathbf{A}$ be defined by* (39). *The operator $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathcal{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, then the index of this operator is*

$$\mathrm{Ind}\mathbf{A} = -\operatorname{ind}\det \mathcal{A}_p.$$

If $\mathcal{A}_p(\omega)$ is the symbol of an operator $\mathbf{A}$ in (39), then the set $\mathcal{R}(\mathcal{A}_p) := \{\mathcal{A}_p(\omega) \in \mathbb{C} : \omega \in \mathfrak{R}\}$ coincides with the essential spectrum of $\mathbf{A}$. Recall that the essential spectrum $\sigma_{ess}(\mathbf{A})$ of a bounded operator $\mathbf{A}$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $\mathbf{A} - \lambda I$ is not Fredholm in $\mathbb{L}_p(\mathbb{R}^+)$ or, equivalently, the coset $[\mathbf{A} - \lambda I]$ is

not invertible in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Then, due to Banach theorem, the essential norm $\|\!|\mathbf{A}|\!\|$ of the operator $\mathbf{A}$ can be estimated as follows

$$\sup_{\omega\in\omega}|\mathcal{A}_p(\omega)| \leqslant \|\!|\mathbf{A}|\!\| := \inf_{\mathbf{T}\in\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))}\big\|(\mathbf{A}+\mathbf{T}) \mid \mathcal{L}(\mathbb{L}_p(\mathbb{R}^+))\big\|. \tag{42}$$

The inequality (42) enables one to extend continuously the symbol map (40)

$$[\mathbf{A}] \longrightarrow \mathcal{A}_p(\omega), \quad [\mathbf{A}] \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$$

on the whole Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$. Now, applying Theorem 4.2 and a standard methods, cf. [17, Theorem 3.2], one can derive the following result.

Corollary 4.3 ([18, Corollary 3.15]) *Let $1 < p < \infty$ and $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)$. The operator $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathcal{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, then*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\mathcal{A}_p.$$

5 Fredholm properties of Mellin Convolution Operators in the Bessel Potential Spaces.

As it was already mentioned, the primary aim of the present paper is to study Fredholm properties and the invertibility of Mellin convolution operators $\mathfrak{M}^0_a$ acting in Bessel potential spaces, namely,

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+).$$

The symbols of these operators are $N\times N$ matrix functions $a \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, continuous on the real axis $\mathbb{R}$ with the only possible jump at infinity.

Theorem 5.1 *Let $s \in \mathbb{R}$ and $1 < p < \infty$.*

1. *If the conditions of Theorem 3.2 hold, then the Mellin convolution operator $\mathbf{K}^1_c$,*

$$\mathbf{K}^1_c \;:\; \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+) \tag{43}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c W_{g^s_{-c\gamma,\gamma}} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+),$$

where $c^{-s} = |c|^{-s}e^{-is\arg c}$ and the function $g^s_{-c\gamma,\gamma}$ is defined in (37).

2. *If conditions of Corollary 3.3 hold, the Mellin convolution operator between Bessel potential spaces* (43) *is lifted to the equivalent operator*

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} &= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{g^s_{-c\gamma_0,\gamma}}\\ &= c^{-s}\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\gamma_0,\gamma}} + \mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+),\end{aligned}$$

where $\mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ *is a compact operator.*

Proof. By Theorem 2.3, using the lifting procedure, one obtains the following equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+).$$

In order to proceed, we need two formulae

$$\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma} = W_{g^s_{-c\gamma,\gamma}}, \qquad W_{g^s_{-\gamma,-\gamma_0}}W_{g^s_{-c\gamma_0,\gamma}} = W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\gamma_0,\gamma}}. \tag{44}$$

The first relation holds because, by the conditions of Theorem 3.2, $0 < \arg\gamma < \pi$ and the second one holds because $g^s_{-\gamma,-\gamma_0}(\xi)$ has a smooth, uniformly bounded analytic extension in the complex lower half plane.

If conditions of Theorem 3.2 are satisfied, we use the relations (22), (44). Thus

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c W_{g^s_{-c\gamma,\gamma}}.$$

If conditions of Corollary 3.3 hold, we successively apply formulae (36), (38), both formulae (44), so that

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} &= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma}\\ &= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{g^s_{-c\gamma_0,\gamma}} = c^{-s}\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}}W_{g^s_{-c\gamma_0,\gamma}} + \mathbf{T},\end{aligned}$$

which completes the proof. ∎

Remark 2 *The operator* $\mathbf{K}^1_1$ *is the Hilbert transform* $\mathbf{K}^1_1 = -\pi i S_{\mathbb{R}^+} = \pi i W_{\mathrm{sign}}$ *and does not satisfy the condition* $\arg c \neq 0$ *of Theorem 5.1. As already emphasized in Theorem 2.3, this case is essentially different. Considered as acting between the Bessel potential spaces* (43), $\mathbf{K}^1_1$ *is lifted to the equivalent Fourier convolution operator*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_1\mathbf{\Lambda}^{-s}_{\gamma} = W_{\pi i g^s_{-\gamma,\gamma}\mathrm{sign}} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+),$$

cf. Theorem 2.3.

Theorem 5.2 *Let $c_j, d_j \in \mathbb{C}$, $-\pi \leqslant \arg c_j < \pi$ $\arg c_j \neq 0$, for $j = 1, \ldots, n$, $0 < \arg\gamma < \pi$, $-\pi < \arg(c_j\gamma) < 0$ for $j = 1, \ldots, m$ and $0 < \arg(c_j\gamma) < \pi$ for $j = m+1, \ldots, n$. The Mellin convolution operator $\mathbf{A}$,*

$$\mathbf{A} = \sum_{j=1}^{n} d_j \mathbf{K}^1_{c_j} \ : \ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+),$$

is lifted to the equivalent operator

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} &= \sum_{j=0}^{m} d_j c_j^{-s} \mathbf{K}^1_{c_j} W_{g^s_{-c_j\gamma,-\gamma}} + \sum_{j=m+1}^{n} d_j c_j^{-s} W_{g^s_{-\gamma,-\gamma_j}} \mathbf{K}^1_{c_j} W_{g^s_{-c_j\gamma_j,\gamma}} \\ &= \sum_{j=0}^{m} d_j c_j^{-s} \mathbf{K}^1_{c_j} W_{g^s_{-c_j\gamma,\gamma}} + \sum_{j=m+1}^{n} d_j c_j^{-s} \mathbf{K}^1_{c_j} W_{g^s_{-\gamma,-\gamma_j} g^s_{-c_j\gamma_j,\gamma}} + \mathbf{T} \end{aligned} \tag{45}$$

in the $\mathbb{L}_p(\mathbb{R}^+)$ space, where $c^{-s} = |c|^{-s} e^{-is\arg c}$ and γ_j are such that $0 < \arg\gamma_j < \pi$, $-\pi < \arg(c_j\,\gamma_j) < 0$ for $j = m+1, \ldots, n$. $\mathbf{T} \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator.

Proof. The proof is a direct consequence of Theorem 5.1. ■

Theorem 5.3 *Let $s \in \mathbb{R}$ and $1 < p < \infty$. If conditions of Theorem 3.2 hold, then the Mellin convolution operator $\mathbf{K}^2_c$,*

$$\mathbf{K}^2_c \ : \ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+) \tag{46}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right] W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\, c^{-s}\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}} \mathbf{\Lambda}^{-1}_{\gamma} \tag{47}$$

in $\mathbb{L}_p(\mathbb{R}^+)$ space, where $c^{-s} = |c|^{-s}e^{-is\arg c}$, the function $g^s_{-c\gamma,\gamma}$ is defined in (37), and the last summand in (47), namely, the operator

$$\mathbf{T} := s\,\gamma\, c^{-s}\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}} \mathbf{\Lambda}^{-1}_{\gamma} \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+), \tag{48}$$

is compact. Moreover, if conditions of Corollary 3.3 hold, the Mellin convolution operator $\mathbf{K}^2_c$ between Bessel potential spaces (46) is lifted to the equivalent operator

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_{\gamma} =& c^{-s} W_{g^s_{-\gamma,-\gamma_0}} \left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right] W_{g^s_{-c\gamma_0,\gamma}} \\ &+ s\,\gamma\, c^{-s} W_{g^s_{-\gamma,-\gamma_0}} \mathbf{K}^1_c W_{g^{s-1}_{-c\gamma_0,\gamma}} \mathbf{\Lambda}^{-1}_{\gamma} \\ =& c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right] W_{g^s_{-\gamma,-\gamma_0} g^s_{-c\,\gamma_0,\gamma}} + \mathbf{T}_0 \end{aligned} \tag{49}$$

in $\mathbb{L}_p(\mathbb{R}^+)$ space, and the operator $\mathbf{T}_0 \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is compact.

Proof. If the conditions of Theorem 3.2 are satisfied, then $\operatorname{Im}\gamma > 0$ and $\operatorname{Im} c\,\gamma < 0$. Hence

$$\frac{1}{(t-c)^2} = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\left[\frac{1}{t-c-\varepsilon i} - \frac{1}{t-c+\varepsilon i}\right]$$

and we have

$$\begin{aligned}
\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_{\gamma} &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\mathbf{\Lambda}^s_{-\gamma}\left[\mathbf{K}^1_{c+\varepsilon i} - \mathbf{K}^1_{c-\varepsilon i}\right]\mathbf{\Lambda}^{-s}_{\gamma}\\
&= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\left[(c+\varepsilon i)^{-s}\mathbf{K}^1_{c+\varepsilon i}\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma} - (c-\varepsilon i)^{-s}\mathbf{K}^1_{c-\varepsilon i}\mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\right]\mathbf{\Lambda}^{-s}_{\gamma}\\
&= \lim_{\varepsilon\to 0}\left\{\frac{(c+\varepsilon i)^{-s} - (c-\varepsilon i)^{-s}}{2\varepsilon i}\mathbf{K}^1_{c+\varepsilon i}\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma}\right.\\
&\quad -(c-\varepsilon i)^{-s}\frac{1}{2\varepsilon i}\left[\mathbf{K}^1_{c+\varepsilon i} - \mathbf{K}^1_{c-\varepsilon i}\right]\mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\\
&\quad \left. -(c-\varepsilon i)^{-s}\mathbf{K}^1_{c-\varepsilon i}\frac{1}{2\varepsilon i}\left[\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma} - \mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\right]\right\}\mathbf{\Lambda}^{-s}_{\gamma}\\
&= -s\,c^{-s-1}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\mathbf{\Lambda}^{-s}_{\gamma} + c^{-s}\mathbf{K}^2_c\mathbf{\Lambda}^s_{-c\,\gamma}\mathbf{\Lambda}^{-s}_{\gamma}\\
&\quad + c^{-s}\mathbf{K}^1_c\lim_{\varepsilon\to 0}\mathcal{F}^{-1}\frac{(\xi - c\,\gamma - \varepsilon\gamma i)^s - (\xi - c\,\gamma + \varepsilon\gamma i)^s}{2\varepsilon i}\mathcal{F}\mathbf{\Lambda}^{-s}_{\gamma}\\
&= c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^{s-1}_{-c\,\gamma}\mathbf{\Lambda}^{-s}_{\gamma}\\
&= c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}}\mathbf{\Lambda}^{-1}_{\gamma}.
\end{aligned}$$

Thus formula (47) is proved. To verify the compactness of the operator $\mathbf{T}$ in (48), let us rewrite it as follows

$$\begin{aligned}
\mathbf{T} &= s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}}\mathbf{\Lambda}^{-1}_{\gamma}\\
&= s\,\gamma\,c^{-s}(1-h)\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}}\mathbf{\Lambda}^{-1}_{\gamma} + s\,\gamma\,c^{-s}h\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}}\mathbf{\Lambda}^{-1}_{\gamma}, \qquad (50)
\end{aligned}$$

where $h \in C^\infty(\mathbb{R})^+$ is a smooth function having a compact support and equal to 1 in a neighborhood of 0. Since $1-h(t)$ vanishes in the neighbourhood of 0, the operator $(1-h)\mathbf{K}^1_c$ has a smooth kernel and is compact in $\mathbb{L}_p(\mathbb{R}^+)$. The second summand in (50) is compact since h commutes with the Mellin $\mathbf{K}^1_c$ and Fourier $W_{g^{s-1}_{-c\gamma,\gamma}}$ convolutions modulo compact operators, i.e.

$$s\,\gamma\,c^{-s}h\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}}\mathbf{\Lambda}^{-1}_{\gamma} = s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}}h\mathbf{\Lambda}^{-1}_{\gamma} + \mathbf{T}_1,$$

where $\mathbf{T}_1$ is compact in $\mathbb{L}_p(\mathbb{R}^+)$ (see Proposition 4.1 and [15, Lemma 7.4], [17, Lemma 1.2]). Note, that due to the Sobolev embedding theorem, the operator

$h\mathbf{\Lambda}_\gamma^{-1}$ is also compact in $\mathbb{L}_p(\mathbb{R}^+)$, because $\operatorname{supp} h$ is compact. Finally, formula (49) can be derived from (47) similarly to considerations of Theorem 5.1. ■

Remark 3 *The operators $\mathbf{K}_c^n$, $n = 3, 4, \ldots$, can be treated analogously to the approach of Corollary 5.3. Indeed, let us represent the operator $\mathbf{K}_c^n$ in the form*

$$\mathbf{K}_c^n\varphi = \lim_{\varepsilon\to 0}\mathbf{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\varphi, \qquad \forall\,\varphi \in \widetilde{\mathbb{H}}_p^r(\mathbb{R}^+),$$

where

$$\begin{aligned}
\mathbf{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\varphi(t) &:= \int_0^\infty \mathcal{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\left(\frac{t}{\tau}\right)\varphi(\tau)\frac{d\tau}{\tau} = \sum_{j=1}^n d_j(\varepsilon)\mathbf{K}_{c_{j,\varepsilon}}^1\varphi(t),\\
\mathcal{K}_{c_{1,\varepsilon},\ldots,c_{m,\varepsilon}}(t) &:= \frac{1}{(t-c_{1,\varepsilon})\cdots(t-c_{n,\varepsilon})} = \sum_{j=1}^n \frac{d_j(\varepsilon)}{t-c_{j,\varepsilon}},\\
c_{j,\varepsilon} &= c(1+\varepsilon e^{i\omega_j}),\ \omega_j\in(-\pi,\pi),\ \arg c_{j,\varepsilon},\ \arg(c_{j,\varepsilon}\,\gamma)\neq 0,\ j=1,\ldots,m.
\end{aligned} \tag{51}$$

Since $n \in \{3, 4, \ldots\}$ the argument $\arg c$ does not vanish. Hence, the points $\omega_1, \ldots, \omega_n \in (-\pi, \pi]$ are pairwise different, i.e., $\omega_j \neq \omega_k$ for $j \neq k$. By equating the numerators in the formula (51) we find the coefficients $d_1(\varepsilon), \ldots, d_{n-1}(\varepsilon)$.

Note that the operators $\mathbf{K}_c^3, \mathbf{K}_c^4, \ldots$ appear rather rarely in applications. Therefore, in this work exact formulae are given in the case of the operators $\mathbf{K}_c^1$ and $\mathbf{K}_c^2$ only.

Assume that $a_0, \ldots, a_n, b_1, \ldots, b_n \in C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\})$, $c_1, \ldots, c_n \in \mathbb{C}$ and consider the model operator $\mathbf{A} : \mathbb{H}_p^s(\mathbb{R}^+) \to \mathbb{H}_p^s(\mathbb{R}^+)$,

$$\mathbf{A} := d_0 I + W_{a_0} + \sum_{j=1}^n W_{a_j}\mathbf{K}_{c_j}^1 W_{b_j}, \tag{52}$$

comprising the identity I, Fourier $W_{a_0}, \ldots, W_{a_n}$, $W_{b_1}, \ldots, W_{b_n}$ and Mellin $\mathbf{K}_{c_1}^1, \ldots, \mathbf{K}_{c_n}^1$ convolution operators. In order to ensure proper mapping properties of the operator $\mathbf{A} : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \to \mathbb{H}_p^s(\mathbb{R}^+)$, we additionally assume that if $s \leq 1/p - 1$ or $s \geq 1/p$, then the functions $a_1(\xi), \ldots, a_n(\xi)$ and $b_1(\xi), \ldots, b_n(\xi)$ have bounded analytic extensions in the lower $\operatorname{Im}\xi < 0$ and the upper $\operatorname{Im}\xi > 0$ half planes, correspondingly.

If $1/p - 1 < s < 1/p$, then the spaces $\widetilde{\mathbb{H}}_p^s(\mathbb{R}^+)$ and $\mathbb{H}_p^s(\mathbb{R}^+)$ coincide (can be identified) and the analytic extendability assumption are not needed. However, we do not consider this situation here since it requires a special treatment. Besides, it does not appear in applications.

Now we can describe the symbol $\mathcal{A}_p^s$ of the model operator $\mathbf{A}$. According to the formulae (40) and (41) one has

$$\mathcal{A}_p^s(\omega) := d_0 \mathcal{I}_p^s(\omega) + \mathcal{W}_{a_0,p}^s(\omega) + \sum_{j=1}^{n} \mathcal{W}_{a_j,p}^0(\omega) \mathcal{K}_{c_j,p}^{1,s}(\omega) \mathcal{W}_{b_j,p}^0(\omega), \tag{53}$$

where the symbols $\mathcal{I}_p^s(\omega)$, $\mathcal{W}_{a,p}^0(\omega)$, $\mathcal{W}_{a,p}^s(\omega)$ and $\mathcal{K}_{c_j,p}^{1,s}(\omega)$ have the form

$$\mathcal{I}_p^s(\omega) := \begin{cases} g_{-\gamma,\gamma,p}^s(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma}_1, \\ \left(\dfrac{\eta-\gamma}{\eta+\gamma}\right)^{\mp s}, & \omega = (+\infty,\eta) \in \Gamma_2^{\pm}, \\ e^{\pi s i}, & \omega = (\xi,0) \in \overline{\Gamma}_3, \end{cases} \tag{54a}$$

$$\mathcal{W}_{a,p}^0(\omega) := \begin{cases} a_p(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma}_1, \\ a(\mp\eta), & \omega = (+\infty,\eta) \in \Gamma_2^{\pm}, \\ a_p(0,\xi), & \omega = (\xi,0) \in \overline{\Gamma}_3, \end{cases} \tag{54b}$$

$$\mathcal{W}_{a,p}^s(\omega) := \begin{cases} a_p^s(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma}_1, \\ a(\mp\eta)\left(\dfrac{\eta-\gamma}{\eta+\gamma}\right)^{\mp s}, & \omega = (+\infty,\eta) \in \Gamma_2^{\pm}, \\ e^{\pi s i} a_p(0,\xi), & \omega = (\xi,0) \in \overline{\Gamma}_3, \end{cases} \tag{54c}$$

$$\mathcal{K}_{c,p}^{1,s}(\omega) := \begin{cases} \dfrac{c^{-s}(-c)^{\frac{1}{p}-i\xi-1}}{\sin\pi(\frac{1}{p}-i\xi)}, & \omega = (\xi,\infty) \in \overline{\Gamma}_1,, \\ 0, & \omega = (\pm\infty,\eta) \in \Gamma_2^{\pm}, \\ \dfrac{c^{-s}(-c)^{\frac{1}{p}+s-i\xi-1}}{\sin\pi(\frac{1}{p}-i\xi)}, & \omega = (\xi,0) \in \overline{\Gamma}_3, \qquad \text{for} \quad 0 < |\arg(c\,\gamma)| < \pi, \end{cases} \tag{54d}$$

$$a_p^s(\infty,\xi) := \frac{e^{2\pi s i}a(\infty) + a(-\infty)}{2} + \frac{e^{2\pi s i}a(\infty) - a(-\infty)}{2i} \cot\pi\Big(\frac{1}{p} - i\xi\Big),$$

$$a_p(x,\xi) := \frac{a(x+0) + a(x-0)}{2} + \frac{a(x+0) - a(x-0)}{2i} \cot\pi\Big(\frac{1}{p} - i\xi\Big), \quad x = 0, \infty,$$

$$g_{-\gamma,\gamma,p}^s(\infty,\xi) := \frac{e^{2\pi s i} + 1}{2} + \frac{e^{2\pi s i} - 1}{2i} \cot\pi\Big(\frac{1}{p} - i\xi\Big) = e^{\pi s i} \frac{\sin\pi\Big(\frac{1}{p} + s - i\xi\Big)}{\sin\pi\Big(\frac{1}{p} - i\xi\Big)},$$

$$\xi \in \mathbb{R}, \quad \eta \in \mathbb{R}^+,$$

where

$$-\pi \leqslant \arg c < \pi, \quad \arg c \neq 0, \quad -\pi < \arg(c\,\gamma_0) < 0, \quad 0 < \arg\gamma, \arg\gamma_0 < \pi,$$

and $c^s = |c|^s e^{is\arg c}$, $(-c)^\delta = |c|^\delta e^{-i\delta\arg c}$ for $c, \delta \in \mathbb{C}$.

In the case where $a(-\infty) = 1$ and $a(+\infty) = e^{2\pi\alpha i}$ the symbol $a^s_p(\infty,\xi)$ takes the form

$$a^s_p(\infty,\xi) = e^{\pi(s+\alpha)i}\frac{\sin\pi\left(\frac{1}{p}+s+\alpha-i\xi\right)}{\cos\pi\left(\frac{1}{p}-i\xi\right)}. \tag{54e}$$

Note, that the Mellin convolution operator $\mathbf{K}^1_{-1}$,

$$\mathbf{K}^1_{-1}\varphi(t) := \frac{1}{\pi}\int\limits_0^\infty \frac{\varphi(\tau)\,d\tau}{t+\tau} = \mathfrak{M}^0_{\mathcal{M}_{\frac{1}{p}}\mathcal{K}^1_{-1}}, \quad \mathcal{M}_{\frac{1}{p}}\mathcal{K}^1_{-1}(\xi) = \frac{1}{\sin\pi\left(\frac{1}{p}-i\xi\right)},$$

which often appears in applications, has a rather simple symbol if considered in the Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^+)$. Thus using formula (54d) with $c=-1$, one obtains

$$\mathcal{K}^{1,s}_{-1,p}(\omega) := \begin{cases} \dfrac{e^{\pi si}}{\sin\pi(\beta-i\xi)}, & \omega = (\xi,\infty) \in \overline{\Gamma}_1 \cup \overline{\Gamma}_3, \\ 0, & \omega = (\pm\infty,\eta) \in \Gamma^\pm_2, . \end{cases}$$

Theorem 5.4 *Let $1 < p < \infty$, $s \in \mathbb{R}$. The operator*

$$\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+) \tag{55}$$

defined in (52) *is Fredholm if and only if its symbol $\mathcal{A}^s_p(\omega)$ described by the relations* (53), (54a)–(54e)*, is elliptic. If $\mathbf{A}$ is Fredholm, then*

$$\mathrm{Ind}\mathbf{A} = -\mathrm{ind}\det\mathcal{A}^s_p.$$

Proof. Let $c_j, d_j \in \mathbb{C}$, $-\pi \leqslant \arg c_j < \pi$ $\arg c_j \neq 0$, for $j = 1,\ldots,n$. Lifting $\mathbf{A}$ up to an operator on the space $\mathbb{L}_p(\mathbb{R}^+)$ we get

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma = d_0\mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_\gamma + \mathbf{\Lambda}^s_{-\gamma}W_{a_0}\mathbf{\Lambda}^{-s}_\gamma + \sum_{j=1}^n W_{a_j}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_{c_j}\mathbf{\Lambda}^{-s}_\gamma W_{b_j}, \tag{56}$$

where $c^{-s} = |c|^{-s}e^{-is\arg c}$ and γ is such that $0 < \arg\gamma < \pi$, $-\pi < \arg(c_j\,\gamma) < 0$ for all $j = m+1,\ldots,n$.

In (56) we used special properties of convolution operators, namely,

$$\mathbf{\Lambda}^s_{-\gamma}W_{a_j} = W_{a_j}\mathbf{\Lambda}^s_{-\gamma}, \quad W_{b_j}\mathbf{\Lambda}^s_\gamma = \mathbf{\Lambda}^s_\gamma W_{b_j}, \quad \mathbf{\Lambda}^{\mp s}_{\pm\gamma} = W_{\lambda^{\mp s}_{\pm\gamma}},$$

which follows from the analytic extendability of the functions $\lambda^s_{-\gamma}, a_1(\xi), \ldots, a_n(\xi)$ and $\lambda^{-s}_{\gamma}, b_1(\xi), \ldots, b_n(\xi)$ into the lower $\operatorname{Im}\xi < 0$ and upper $\operatorname{Im}\xi > 0$ half planes, respectively.

The model operators I, W_a and $\mathbf{K}^1_c$ lifted to the space $\mathbb{L}_p(\mathbb{R}^+)$ have the form

$$\mathbf{\Lambda}^s_\gamma I \mathbf{\Lambda}^{-s}_\gamma = W_{g^s_{-\gamma,\gamma}}, \qquad \mathbf{\Lambda}^s_\gamma W_a \mathbf{\Lambda}^{-s}_\gamma = W_{a g^s_{-\gamma,\gamma}},$$

$$\mathbf{\Lambda}^s_\gamma \mathbf{K}^1_c \mathbf{\Lambda}^{-s}_\gamma = \begin{cases} c^{-s}\mathbf{K}^1_c W_{g^s_{-c\,\gamma,\gamma}} & \text{if} \quad -\pi < \arg(c\,\gamma) < 0, \\[2ex] c^{-s}\mathbf{K}^1_c W_{g^s_{-\,\gamma,-\gamma_0} g^s_{-c\,\gamma_0,\gamma}} + \mathbf{T}, & \text{if} \quad \begin{array}{c} 0 < \arg(c\,\gamma) < \pi, \\ -\pi < \arg(c\,\gamma_0)| < 0, \end{array} \end{cases} \tag{57}$$

where $\mathbf{T}$ is a compact operator. Here, as above, $-\pi \leqslant \arg c < \pi$, $\arg c \neq 0$, $0 < \arg \gamma < \pi$, $0 < \arg \gamma_0 < \pi$ and either $-\pi < \arg(c\,\gamma) < 0$ or, if $-\pi < \arg(c\,\gamma) < 0$, then $-\pi < \arg(c\,\gamma_0)| < 0$. Recall that $c^{-s} = |c|^{-s} e^{-is \arg c}$.

Therefore, the operator $\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma$ in (56) can be rewritten as follows

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma = d_0 W_{g^s_{-\gamma,\gamma}} + W_{a_0 g^s_{\gamma,\gamma}} + \sum_{j=1}^{m} c_j^{-s} W_{a_j} \mathbf{K}^1_{c_j} W_{g^s_{-c_j\gamma,-\gamma}} W_{b_j} \\ + \sum_{j=m+1}^{n} c_j^{-s} W_{a_j} \mathbf{K}^1_{c_j} W_{g^s_{-\gamma,-\gamma_j} g^s_{-c_j\gamma_j,\gamma}} W_{b_j} + \mathbf{T} \ :\ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \end{aligned} \tag{58}$$

where $\mathbf{T}$ is a compact operator and we ignore it when writing the symbol of $\mathbf{A}$.

Now we define the symbol of the initial operator $\mathbf{A} \,:\, \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$ of (52) as the symbol of the corresponding lifted operator $\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_\gamma : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ of (58).

To write the symbol of the lifted operator in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ let us first find the limits of involved functions (symbols). The function $g^s_{-\gamma,\gamma} \in C(\mathbb{R})$ is continuous on $\mathbb{R}$, but has different limits at the infinity, viz.,

$$g^s_{-\gamma,\gamma}(-\infty) = 1, \quad g^s_{-\gamma,\gamma}(+\infty) = e^{2\pi s i}, \qquad g^s_{-\gamma,\gamma}(0) = e^{\pi s i}, \tag{59}$$

while the functions $g^s_{-\gamma,-\gamma_0},\ g^s_{-c\gamma,\gamma}, g^s_{-c\gamma_0,\gamma} \in C(\mathbb{R})$ are continuous on $\mathbb{R}$ including the infinity. Thus

$$\begin{aligned} g^s_{-c\,\gamma,\gamma}(\pm\infty) = g^s_{-\gamma,-\gamma_0}(\pm\infty) = g^s_{-c\,\gamma_0,\gamma}(\pm\infty) = 1, \\ g^s_{-\gamma,-\gamma_0}(0) g^s_{-c\,\gamma_0,\gamma}(0) = \left(\frac{-\gamma}{-\gamma_0}\right)^s \left(\frac{-c\gamma_0}{\gamma}\right)^s = (-c)^s\,, \\ g^s_{-c\,\gamma,\gamma}(0) = (-c)^s \quad \text{if} \quad -\pi \leqslant \arg c < \pi, \quad \arg c \neq 0. \end{aligned} \tag{60}$$

In the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$, the symbols of the first two operators in (58), are written according the formulae (40)–(41) by taking into account the equalities

(59) and (60). The symbols of these operators have, respectively, the form (54a) and (54c).

For the operators $W_{a_1}, \dots, W_{a_n}$ and $W_{b_1}, \dots, W_{b_n}$ we can use the formulae (40)–(41) and write their symbols in the form (54b).

The lifted Mellin convolution operators

$$\mathbf{\Lambda}^s_\gamma \mathbf{K}^1_{c_j} \mathbf{\Lambda}^{-s}_\gamma \ : \ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow, \mathbb{L}_p(\mathbb{R}^+)$$

comprise both the Fourier convolution operators $W_{g^s_{-c_j\,\gamma_0,\gamma}}$ and $W_{g^s_{-\gamma,-\gamma_0} g^s_{-c_j\,\gamma_0,\gamma}}$ and the Mellin convolution operators $\mathbf{K}^1_{c_j} = \mathfrak{M}^0_{\mathcal{K}^1_{c_j,p}(\xi)}$, with the symbol $\mathcal{K}^1_{c_j,p}(\xi) := \mathcal{M}_{1/p}\mathcal{K}^1_{c_j}(\xi)$ defined in (9) and (10). The symbol of the operators $\mathbf{\Lambda}^s_\gamma \mathbf{K}^1_{c_j} \mathbf{\Lambda}^{-s}_\gamma$ from (57) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ is found according formulae (40)–(41), has the form (54d) and is declared the symbol of $\mathbf{K}^1_{c_j} \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$. The symbols of Fourier convolution factors $W_{g^s_{-c_j\,\gamma_0,\gamma}}$ and $W_{g^s_{-\gamma,-\gamma_0} g^s_{-c_j\,\gamma_0,\gamma}}$, which contribute the symbol of $\mathbf{K}^1_{c_j} = \mathfrak{M}^0_{\mathcal{K}^1_{c_j,p}}$ are written again according formulae (40)–(41) by taking into account the equalities (59) and (60).

To the lifted operator applies Theorem 4.2 and gives the result formulated in Theorem 5.4. ■

In the proof of the foregoing Theorem 5.4, a local principle is used. As a byproduct, a result which itself is important in applications is obtained. We formulate it separately as a corollary. Note that the definition of the local invertibility and a short introduction to a local principle can be found in [25, 34].

Corollary 5.5 *Let $1 < p < \infty$, $s \in \mathbb{R}$. The operator*

$$\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+),$$

defined in (52)*, is locally invertible at* $0 \in \mathbb{R}^+$ *if and only if its symbol* $\mathcal{A}^s_p(\omega)$*, defined in* (53)*,* (54a)–(54e)*, is elliptic on* Γ_1*, i.e.*

$$\inf_{\omega\in\Gamma_1} \left|\det \mathcal{A}^s_p(\omega)\right| = \inf_{\xi\in\mathbb{R}} \left|\det \mathcal{A}^s_p(\xi,\infty)\right| > 0.$$

The next results are concerned with the operators acting in the Sobolev–Slobodeckij (Besov) spaces. For the definition of the corresponding spaces $\mathbb{W}^s_p(\Omega) = \mathbb{B}^s_{p,p}(\Omega)$, $\widetilde{\mathbb{W}}^s_p(\Omega) = \widetilde{\mathbb{B}}^s_{p,p}(\Omega)$ for an arbitrary domain $\Omega \subset \mathbb{R}^n$, including the semi-axis $\mathbb{R}^+$, we refer the reader to the monograph [36].

Corollary 5.6 *Let $1 < p < \infty$, $s \in \mathbb{R}$. If the operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$, defined in* (52)*, is Fredholm (invertible) for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$, where $-\infty < s_0 < s_1 < \infty$, $1 < p_o < p_1 < \infty$, then the operator*

$$\mathbf{A} : \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{W}^s_p(\mathbb{R}^+), \quad s \in (s_0, s_1), \ \ p \in (p_0, p_1) \tag{61}$$

is Fredholm (invertible) in the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p = \mathbb{B}^s_{p,p}$, *and*

$$\mathrm{Ind}\mathbf{A} = -\mathrm{ind}\det \mathcal{A}^s_p. \tag{62}$$

Proof. Recall that the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p = \mathbb{B}^s_{p,p}$ emerge as the result of interpolation with the real interpolation method between Bessel potential spaces

$$\begin{aligned}
&\left(\mathbb{H}^{s_0}_{p_0}(\Omega), \mathbb{H}^{s_1}_{p_1}(\Omega)\right)_{\theta,p} = \mathbb{W}^s_p(\Omega), \quad s := s_0(1-\theta) + s_1\theta,\\
&\left(\widetilde{\mathbb{H}}^{s_0}_{p_0}(\Omega), \widetilde{\mathbb{H}}^{s_1}_{p_1}(\Omega)\right)_{\theta,p} = \widetilde{\mathbb{W}}^s_p(\Omega), \quad p := \frac{1}{p_0}(1-\theta) + \frac{1}{p_1}\theta, \quad 0 < \theta < 1.
\end{aligned} \tag{63}$$

If $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm (invertible) for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$, it has a regularizer $\mathbf{R}$ (the inverse $\mathbf{A}^{-1} = \mathbf{R}$, respectively), which is bounded in the setting

$$\mathbf{R} : \mathbb{W}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$$

due to the interpolation (63) and

$$\mathbf{RA} = I + \mathbf{T}_1, \quad \mathbf{AR} = I + \mathbf{T}_2,$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and in $\mathbb{H}^s_p(\mathbb{R}^+)$, or $\mathbf{T}_1 = \mathbf{T}_2 = 0$ if $\mathbf{A}$ is invertible.

Due to the Krasnoselskij interpolation theorem (see [36]), $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$ and in $\mathbb{W}^s_p(\mathbb{R}^+)$, respectively for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$ and, therefore, $\mathbf{A}$ in (61) is Fredholm (is invertible, respectively).

The index formulae (62) follows from the embedding properties of the Sobolev–Slobodeckij and Bessel potential spaces by standard well-known arguments. ■

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